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A MATRIX BASED EXPLICIT FORMULATION FOR INVERSE RADIATIVE TRANSFER PROBLEMS

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Abstract – In the present work we present an explicit matrix formulation for the inverse problem of radiative properties estimation. By properly arranging the information related to the boundary conditions and experimental data we are able to estimate the scattering and total extinction coefficients of a one-dimensional homogeneous participating medium. As experimental data we use the intensity of the exit radiation measured at the boundaries of the medium. Test case results are presented.

1. INTRODUCTION

Inverse radiative transfer problems can be formulated either explicitly or implicitly, [6]. Roberty, Silva Neto and co-workers have developed an explicit formulation, the so-called source-detector methodology, [3]. Recently, we have developed another explicit formulation for the inverse problem of estimating the radiative properties of one-dimensional participating media based only on matrix manipulations, [1, 4]. Two matrices are constructed, one for each boundary of the medium. Such matrices are constructed using a proper arrangement of the boundary conditions imposed on the radiative transfer problem and the experimental data on the radiation that leaves the medium under analysis. Only external detectors are considered.

A finite dimensional version of the problem is obtained with the discretization of the angular domain using the concepts of the discrete ordinates method and the expansion of the phase function of anisotropic scattering in a series of Legendre polynomials.

Estimates are then obtained for the absorption and scattering coefficients. From the higher order terms the total extinction coefficient is estimated and from the lower order terms the scattering coefficient is obtained.

The strategies developed for the implementation of the method are described and a few test cases are considered in order to demonstrate the feasibility of the method.

2. EXPLICIT MATRIX FORMULATION

2.1 Direct Problem

Consider the radiative transfer problem in the absorbing and anisotropic scattering plane-parallel gray medium of thickness L represented in Figure 1.

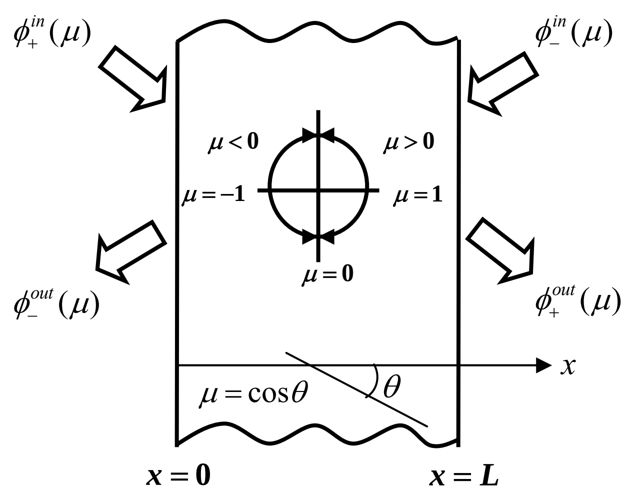


Figure 1. One-dimensional plane-parallel participating medium.

The mathematical description of the radiation interaction with the participating medium is given by the linear version of the Boltzmann equation, which for the case of azimuthal symmetry is written as [1],

$$\mu \frac{\partial \phi(x,\mu)}{\partial x} = -\sigma_t(x)\,\phi(x,\mu) + \sum_{l=0}^{M} \sigma_{sl}(x)\; p_l(\mu) \int_{-1}^{1} \phi(x,\mu')\; p_l(\mu')\, d\mu' , \quad 0 \le x \le L \text{ and } -1 \le \mu \le 1 \tag{1}$$

$$\phi(0,\mu) = \phi_+^{in}(\mu)\,,\; 0 \le \mu \le 1 \tag{2}$$

$$\phi(L,\mu) = \phi_-^{in}(\mu)\,,\; -1 \le \mu \le 1 \tag{3}$$

where $\phi(x,\mu)$ represents the radiation intensity, x is the spatial coordinate, μ is the cosine of the polar angle, σ_t is the total extinction coefficient, σ_s is the scattering coefficient (σ_{sl}, $l = 0,1,\dots,M$, corresponding to the expansion coefficients of the phase function of anisotropic scattering in Legendre polynomials, with a total number of $M+1$ coefficients), p_l represents the normalized Legendre polynomials, and ϕ_+^{in} and ϕ_-^{in} represent the intensity of the incoming radiation at $x = 0$ and $x = L$, respectively.

Equation (1) may be written as

$$\left[T \frac{\partial \phi}{\partial x}\right](x,\mu) = -[A_c \phi](x,\mu) \tag{4}$$

where T is the multiplicative operator, i.e.

$$\phi(x,\mu) \xrightarrow{T} \mu\; \phi(x,\mu) \tag{5}$$

and A_c is the collision operator, i.e

$$\phi(x,\mu) \xrightarrow{A_c} \sigma_t(x)\; \phi(x,\mu) - \sum_{l=0}^{M} \sigma_{sl}(x)\; p_l(\mu) \int_{-1}^{1} \phi(x,\mu') p_l(\mu') d\mu' \tag{6}$$

Considering the discretization of the angular domain represented in Figure 2, and defining

$$\vec{\phi}(x) = \begin{Bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_N(x) \end{Bmatrix} = \begin{Bmatrix} \phi(x,\mu_1) \\ \phi(x,\mu_2) \\ \vdots \\ \phi(x,\mu_N) \end{Bmatrix} \tag{7}$$

$$T = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_N \end{bmatrix} \tag{8}$$

the solution of eqn. (4) is given by

$$\vec{\phi}(x) = \exp\left[-(x - x_0)\; T^{-1}\; A_c\right] \vec{\phi}(x_0) \tag{9}$$

where x_0 represents the location at which the radiation comes into the medium.

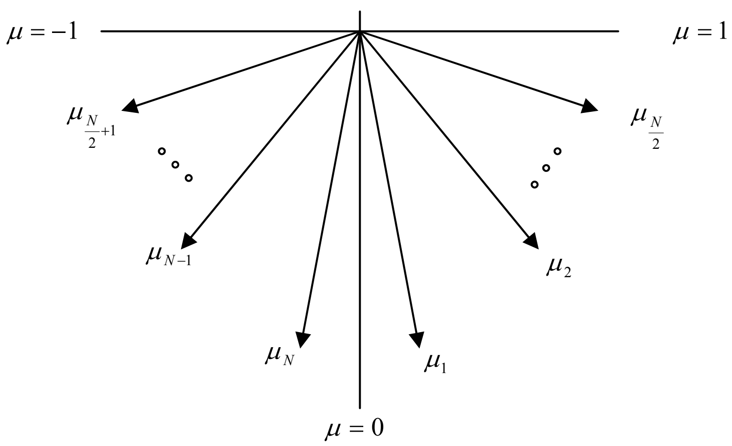


Figure 2. Discretization of the angular domain.

Defining the operator

$$B = \left[T^{-1} A_c \right]^{-1} = A_c^{-1}\, T \tag{10}$$

we may obtain a diagonal matrix T_λ such that [2]

$$B = F\, T_\lambda^{-1}\, F^{-1} \tag{11}$$

From eqns (6) and (7) we obtain

$$A_c = TFT_\lambda F^{-1} \text{ or } T^{-1}A_c = FT_\lambda F^{-1} \tag{12, 13}$$

such that

$$T_\lambda g = \lambda g \tag{14}$$

and λ are the eigenvalues of $T^{-1}A_c$.

2.2 Inverse Problem

We are here interested in estimating the total extinction coefficient σ_t and the coefficients σ_{sl}, $l = 0, 1, \ldots, M$, of the expansion of the anisotropic phase function in Legendre polynomials.

We construct a set of N experiments in which for each experiment i with $i = 1, 2, \ldots N$, we apply the boundary conditions

$$\phi^i(0,\mu) = f_0(\mu)\delta(\mu - \mu_i)\text{, if } 1 \le i \le N/2 \tag{15}$$

$$\phi^i(L,\mu) = f_L(\mu)\ \delta(\mu - \mu_i)\text{, if } N/2 + 1 \le i \le N \tag{16}$$

We then construct the following matrices $\Phi(0)$ and $\Phi(L)$ by properly arranging the available information of the boundary conditions (radiation coming into the medium at $\tau = 0$, with $\mu > 0$, i.e. $f_0(\mu)$ and at $\tau = \tau_0$, with $\mu < 0$, i.e. $f_L(\mu)$) and exit radiation intensities ($\phi_j^i(0)$ for $\mu < 0$, and $\phi_j^i(L)$ for $\mu > 0$, with $i = 1, 2, \ldots N$ and $j = 1, 2, \ldots, N$).

Boundary Condition | Boundary Condition

$$\Phi(0) = \left[\begin{array}{cccc|cccc}
f_0(\mu_1) & 0 & \cdots & 0 & & & & \\
0 & f_0(\mu_2) & \cdots & 0 & & & & \\
\vdots & \vdots & \ddots & \vdots & & 0 & & \\
0 & 0 & \cdots & f_L\left(\mu_{N/2}\right) & & & & \\
\hline
\phi^1_{N/2+1}(0) & \phi^2_{N/2+1}(0) & \cdots & \phi^{N/2}_{N/2+1}(0) & \phi^{N/2+1}_{N/2+1}(0) & \phi^{N/2+2}_{N/2+1}(0) & \cdots & \phi^{N}_{N/2+1}(0) \\
\phi^1_{N/2+2}(0) & \phi^2_{N/2+2}(0) & \cdots & \phi^{N/2}_{N/2+2}(0) & \phi^{N/2+1}_{N/2+2}(0) & \phi^{N/2+2}_{N/2+2}(0) & \cdots & \phi^{N}_{N/2+2}(0) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\phi^1_{N}(0) & \phi^2_{N}(0) & \cdots & \phi^{N/2}_{N}(0) & \phi^{N/2+1}_{N}(0) & \phi^{N/2+2}_{N}(0) & \cdots & \phi^{N}_{N}(0)
\end{array}\right]
\begin{array}{c} \mu > 0 \\ \updownarrow \\ \mu < 0 \end{array} \tag{17}$$

Experimental Data | Experimental Data

Experimental Data | Experimental Data

$$\Phi(L) = \left[\begin{array}{cccc|cccc}
\phi^1_1(L) & \phi^2_1(L) & \cdots & \phi^{N/2}_1(L) & \phi^{N/2+1}_1(L) & \phi^{N/2+2}_1(L) & \cdots & \phi^{N}_1(L) \\
\phi^1_2(L) & \phi^2_2(L) & \cdots & \phi^{N/2}_2(L) & \phi^{N/2+1}_2(L) & \phi^{N/2+2}_2(L) & \cdots & \phi^{N}_2(L) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\phi^1_{N/2}(L) & \phi^2_{N/2}(L) & \cdots & \phi^{N/2}_{N/2}(L) & \phi^{N/2+1}_{N/2}(L) & \phi^{N/2+2}_{N/2}(L) & \cdots & \phi^{N}_{N/2}(L) \\
\hline
& & & & f_L\left(\mu_{N/2+1}\right) & 0 & \cdots & 0 \\
& 0 & & & 0 & f_L\left(\mu_{N/2+2}\right) & \cdots & \\
& & & & \vdots & \vdots & \ddots & \vdots \\
& & & & 0 & 0 & \cdots & f_L(\mu_N)
\end{array}\right]
\begin{array}{c} \mu > 0 \\ \updownarrow \\ \mu < 0 \end{array} \tag{18}$$

Boundary Condition | Boundary Condition

Using $x = L$ and $x_0 = 0$ in eqn. (9), we write

$$\vec{\phi}(L) = \exp\left[-LT^{-1}A_c\right]\vec{\phi}(0) \tag{19}$$

Defining the transmission operator T_r such that

$$\vec{\phi}(L) \xrightarrow{T_r} \exp\left[-LT^{-1}A_c\right]\vec{\phi}(0) \tag{20}$$

we write

$$\Phi(L) = T_r \Phi(0) \tag{21}$$

From eqns (12), (13), (20) and (21) we obtain

$$T_r = \exp\left[-LFT_\lambda F^{-1}\right] \tag{22}$$

which results in

$$T_r = F\exp\left[-LT_\lambda\right]F^{-1} = FDF^{-1} \tag{23}$$

where D is a diagonal matrix.

If $\Phi(0)$ is invertible, we obtain from eqns (21) and (23),

$$\Phi(L)\Phi(0)^{-1} = T_r = FDF^{-1} \tag{24}$$

On defining

$$\ln[\Phi(L)\,\Phi(0)^{-1}] = F\ln DF^{-1} \tag{25}$$

we get from eqns (12), (13) and (24),

$$A_c = -\frac{1}{L}T\ln[\Phi(L)\Phi(0)^{-1}] \tag{26}$$

Using a Gaussian quadrature in the last term of the right hand side of eqn. (1) with the collocation points μ_i, $i = 1,2,...,N$, represented in Figure 2, and the weights of the quadrature a_i, $i = 1,2,...,N$, eqn. (4) is rewritten as

$$T\left[\frac{\partial\vec{\phi}}{\partial x}\right](x) = -\left[A_c W\vec{\phi}\right](x) \tag{27}$$

where

$$W = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_N \end{bmatrix} \tag{28}$$

The elements of matrix A_c are

$$\left[A_c\right]_{ij} = \sigma_t(x)\delta_{ij} + \sum_{l=0}^{M}\sigma_{sl}(x)p_l\left(\mu_i\right)a_j p_l\left(\mu_j\right) \tag{29}$$

such that

$$\left[A_c\vec{\phi}\right]_i = \sum_{j=1}^{N}\left[A_c\right]_{ij}\phi_j = \sigma_t(x)\,\phi_i + \sum_{l=0}^{M}\sigma_{sl}(x)\,p_l(\mu_i)\sum_{j=1}^{N}a_j\,p_l(\mu_j)\,\phi_j \tag{30}$$

Using the Fourier-Legendre series

$$\phi(x,\mu) = \sum_{l=0}^{\infty}\left[\int_{-1}^{1}p_l(\mu')\,\phi(x,\mu')\,d\mu'\right]p_l(\mu) \tag{31}$$

the orthogonality condition

$$\int_{-1}^{1}p_l\left(\mu\right)p_n\left(\mu\right)d\mu = \delta_{ln} \tag{32}$$

and knowing that

$$\int_{-1}^{1}p_l\left(\mu\right)d\mu = 0, \qquad \text{for } l \geq 1 \tag{33}$$

eqn. (30) may be written as

$$\left[A_c\vec{\phi}\right]_i = \sum_{j=1}^{N}\sum_{l=0}^{\infty}\left[\sigma_t(x) - \sigma_{sl}(x)\right]p_l(\mu_i)\,p_l(\mu_j)\,a_j\,\phi_j\text{, with } \sigma_{sl} = 0 \text{ for } l > M \tag{34}$$

Truncating the second summation in eqn. (34) at $l = M_1$, with $M_1 > M$, and defining the matrices

$$P = \begin{bmatrix} p_1(\mu_1) & p_2(\mu_1) & \cdots & p_{M_1}(\mu_1) \\ p_1(\mu_2) & p_2(\mu_2) & \cdots & p_{M_1}(\mu_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(\mu_N) & p_2(\mu_N) & \cdots & p_{M_1}(\mu_N) \end{bmatrix} \tag{35}$$

$$\sigma = \begin{bmatrix} \sigma_t - \sigma_{s0} & 0 & \cdots & 0 \\ 0 & \sigma_t - \sigma_{s1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_t - \sigma_{sM_1} \end{bmatrix} \tag{36}$$

we write

$$A_c\vec{\phi} = P\sigma P^{\mathrm{T}} W\vec{\phi} \tag{37}$$

From eqns (25), (26) and (37) we obtain

$$A_c = P\sigma P^{\mathrm{T}} W = -\frac{1}{L} TF \ln DF^{-1} \tag{38}$$

and therefore

$$\sigma = -\frac{1}{L} P^{\mathrm{T}} TF \ln DF^{-1} PW^{-1} \tag{39}$$

The values of σ_{sl} decrease with the increasing value of l , and also $\sigma_{sl} = 0$ for $l > M$. Therefore, from the higher order terms we obtain the estimated value for σ_t , and we are then able to estimate the values of σ_{sl} for the lower order terms, i.e. $l \le M$.

Using the known boundary conditions and the measured values for the intensity of the exit radiation, properly arranged in matrices $\Phi(0)$ and $\Phi(L)$, matrices F and D are calculated, and then estimates for the unknowns σ_t and σ_{sl} , $l = 1, 2, ..., M$, are directly obtained from eqn. (39).

As the unknowns appear explicitly in the formulation of the inverse problem, this formulation is classified as an explicit one. Observe that the solution of the direct problem is not required.

3. RESULTS AND DISCUSSION

As real experimental data were not available, we have generated synthetic experimental data by adding random noise to the calculated values of the radiation intensity

$$\phi_{experimental} = \phi_{calculated} + r_i\,\sigma\,, \qquad i = 1, 2, ..., N \tag{40}$$

where r_i represents pseudo–random numbers chosen from a normal distribution with zero mean, and σ emulates the standard deviation of the measurement errors.

In order to demonstrate the feasibility of the explicit formulation developed we have chosen a slab of thickness $L = 0.5$ *cm.* As shown in Table 1 four sets of exact values of the scattering coefficients σ_{sl} are considered and for each set three different values are used for the extinction coefficient, namely $\sigma_t = 0.5\ cm^{-1}$, $0.8\ cm^{-1}$ and $1.2\ cm^{-1}$, yielding a total of 12 test cases. Cases A and C represent high and low scattering media, respectively.

In Table 2 are shown the estimated values for the scattering σ_{sl} and extinction σ_t coefficients considering $N = 32$ discrete ordinates for the angular domain (see Figure 2) and four different values for the standard deviation of the measurement errors: $\sigma = 0.0,\ 0.01,\ 0.02$ and 0.03, which correspond respectively to measurement errors up to $0\%,\ 1\%,\ 2\%$ and 3%. These results are also presented in Figures 3-6.

In Table 3 are considered the same test cases but the estimated values were obtained using $N = 16$ instead of $N = 32$.

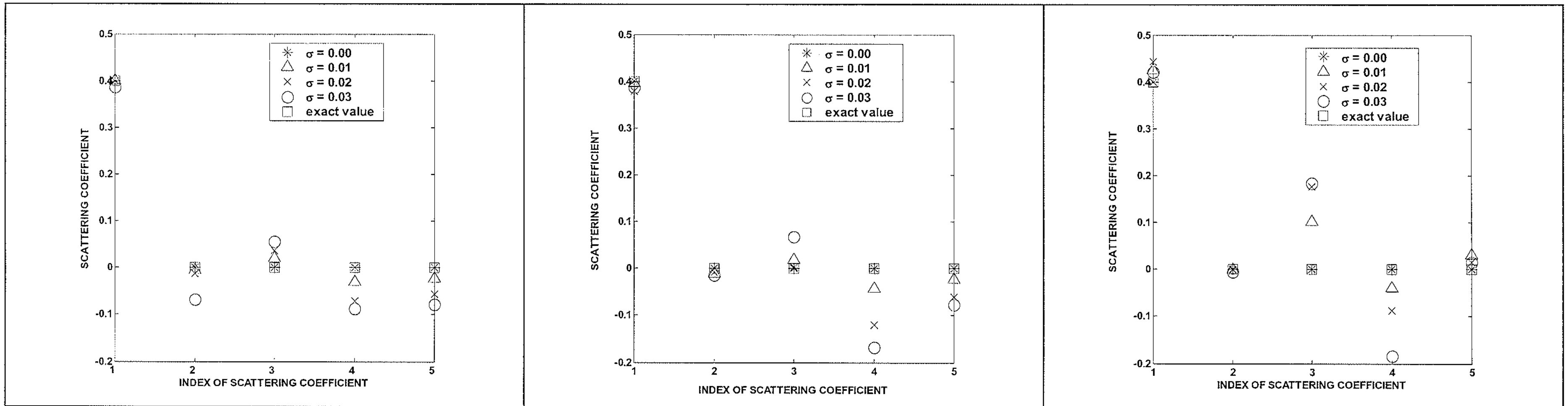
From Tables 2 and 3 we observe that the explicit formulation developed accurately estimates the scattering and extinction coefficients. As observed previously by Silva Neto and Özisik [5], the estimated values for the higher order terms are more affected by the noise in the experimental data. Besides that, the coarser mesh with $N = 16$, yielded in general estimates with smaller percentage errors. However, as shown in Table 4 if N is further reduced the estimates for σ_t are degraded. It seems that a regularization effect is associated with the discretization of the angular domain. Further investigation on this subject must be performed in the future.

Table 1: Exact values of scattering (σ_{sl}) and extinction (σ_t) coefficients $\left[cm^{-1}\right]$.

Test case		Scattering coefficients	Extinction coefficients
1	A	[0.4, 0.0, 0.0, 0.0, 0.0]	0.5
	B		0.8
	C		1.2
2	A	[0.4, 0.2, 0.0, 0.0, 0.0]	0.5
	B		0.8
	C		1.2
3	A	[0.4, 0.2, 0.1, 0.0, 0.0]	0.5
	B		0.8
	C		1.2
4	A	[0.4, 0.2, 0.1, 0.05, 0.0]	0.5
	B		0.8
	C		1.2

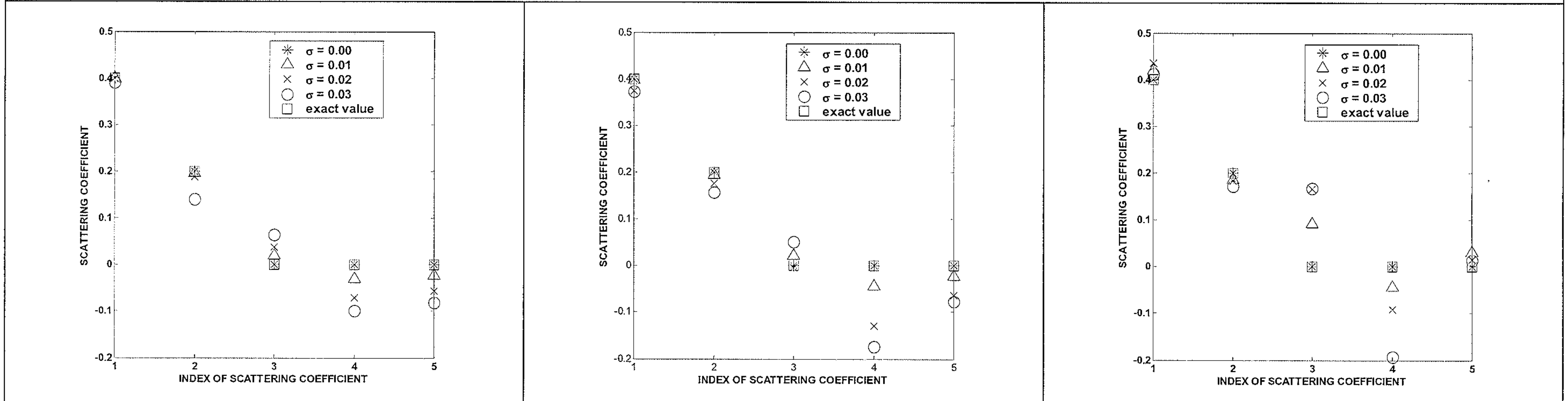
Table 2. Estimated values for the scattering (σ_{sl}) and extinction (σ_t) coefficients $\left[cm^{-1}\right]$ using $N = 32$, and $\sigma = 0.0,\ 0.01,\ 0.02$ and 0.03. $L = 0.5\ cm.$

Test case		Estimated values for σ_{sl}								Estimated values for σ_t							
		σ								σ							
		0.00	% error	0.01	% error	0.02	% error	0.03	% error	0.00	% error	0.01	% error	0.02	% error	0.03	% error
1	A	0.4	0.00	0.4	0.11	0.399	0.25	0.386	3.53	0.5	0.00	0.492	1.51	0.477	4.61	0.472	5.63
	B	0.4	0.00	0.394	1.39	0.379	5.21	0.386	3.61	0.8	0.00	0.795	0.66	0.794	0.82	0.756	5.56
	C	0.4	0.00	0.425	6.15	0.442	10.56	0.420	5.012	1.2	0.00	1.155	3.79	1.122	6.48	1.109	7.56
2	A	0.4	0.00	0.40	0.12	0.399	0.20	0.390	2.55	0.5	0.00	0.493	1.50	0.478	4.59	0.451	8.75
		0.2	0.00	0.195	2.50	0.189	5.72	0.140	29.83								
	B	0.4	0.00	0.397	0.85	0.375	6.34	0.373	6.66	0.8	0.00	0.794	0.72	0.787	1.61	0.754	5.76
		0.2	0.00	0.195	2.76	0.177	11.81	0.157	21.33								
	C	0.4	0.00	0.418	4.52	0.436	8.88	0.411	2.76	1.2	0.00	1.154	3.84	1.122	6.48	1.108	7.68
		0.2	0.00	0.185	7.60	0.184	8.27	0.172	13.94								
3	A	0.4	0.00	0.400	0.11	0.400	0.21	0.393	1.82	0.5	0.00	0.493	1.50	0.477	4.59	0.456	8.75
		0.2	0.00	0.195	2.50	0.188	6.19	0.138	31.04								
		0.1	0.00	0.117	17.10	0.132	31.90	0.155	55.39								
	B	0.4	0.00	0.397	0.75	0.385	3.73	0.364	8.97	0.8	0.00	0.794	0.80	0.785	1.93	0.760	4.99
		0.2	0.00	0.195	2.75	0.187	6.43	0.151	24.52								
		0.1	0.00	0.121	20.65	0.121	20.99	0.127	26.82								
	C	0.4	0.00	0.421	5.22	0.431	7.72	0.440	9.99	1.2	0.00	1.155	3.72	1.122	6.50	1.098	8.50
		0.2	0.00	0.194	2.96	0.175	12.37	0.172	14.03								
		0.1	0.00	0.182	81.85	0.238	137.5	0.284	184.4								
4	A	0.4	0.00	0.40	0.12	0.399	0.23	0.394	1.62	0.5	0.00	0.492	1.58	0.476	4.79	0.457	8.56
		0.2	0.00	0.195	2.49	0.188	6.12	0.135	29.43								
		0.1	0.00	0.117	16.61	0.130	29.76	0.156	56.06								
		0.05	0.00	0.018	63.71	-0.029	158.7	-0.05	202.1								
	B	0.4	0.00	0.397	0.74	0.386	3.59	0.360	9.97	0.8	0.00	0.794	0.81	0.784	2.01	0.759	5.08
		0.2	0.00	0.195	2.75	0.187	6.39	0.147	26.46								
		0.1	0.00	0.121	20.79	0.122	22.14	0.124	24.35								
		0.05	0.00	0.009	83.08	-0.061	221.6	-0.136	371.6								
	C	0.4	0.00	0.421	5.13	0.430	7.47	0.439	9.77	1.2	0.00	1.156	3.71	1.122	6.51	1.098	8.49
		0.2	0.00	0.194	2.96	0.174	13.02	0.171	14.46								
		0.1	0.00	0.181	81.10	0.235	135.4	0.282	182.2								
		0.05	0.00	0.006	87.7	-0.046	192.6	-0.111	321.2								



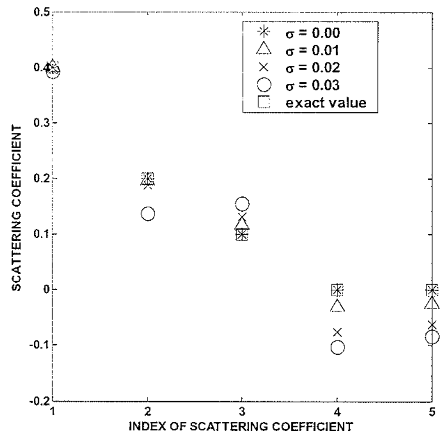
a) Case 1A $(\sigma_t = 0.5)$ | b) Case 1B $(\sigma_t = 0.8)$ | c) Case 1C $(\sigma_t = 1.2)$

Figure 3. Estimated values for the scattering coefficients. Test Case 1 $\left(\sigma_{sl} = [0.4,\ 0.0,\ 0.0,\ 0.0,\ 0.0]\ cm^{-1}\right)$. $L = 0.5\ cm.$ $N = 32$.

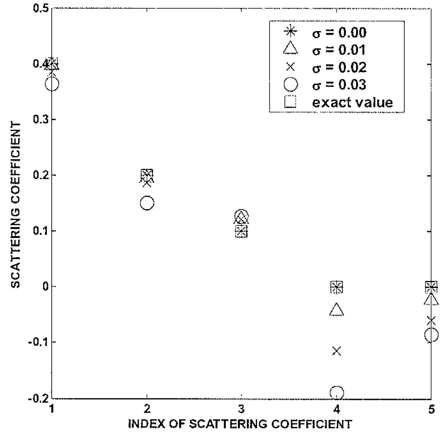


a) Case 2A $(\sigma_t = 0.5)$ | b) Case 2B $(\sigma_t = 0.8)$ | c) Case 2C $(\sigma_t = 1.2)$

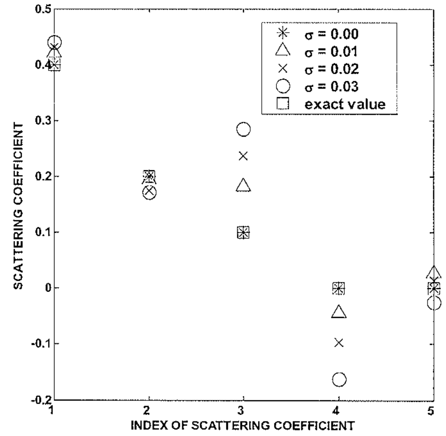
Figure 4. Estimated values for the scattering coefficients. Test Case 2 $\left(\sigma_{sl} = [0.4,\ 0.2,\ 0.0,\ 0.0,\ 0.0]\ cm^{-1}\right)$. $L = 0.5\ cm.$ $N = 32$.



a) Case 3A ($\sigma_t = 0.5$)

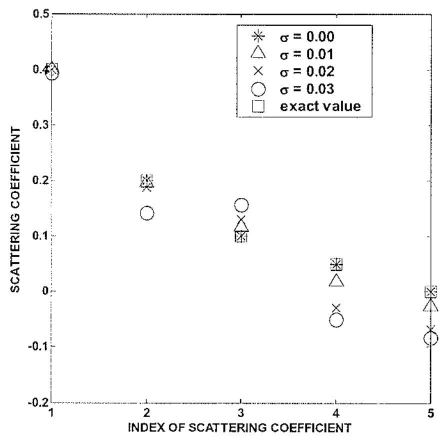


b) Case 3B ($\sigma_t = 0.8$)

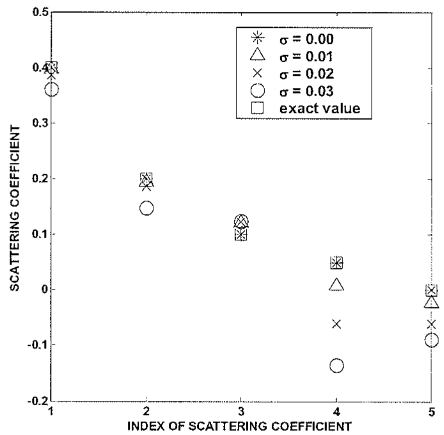


c) Case 3C ($\sigma_t = 1.2$)

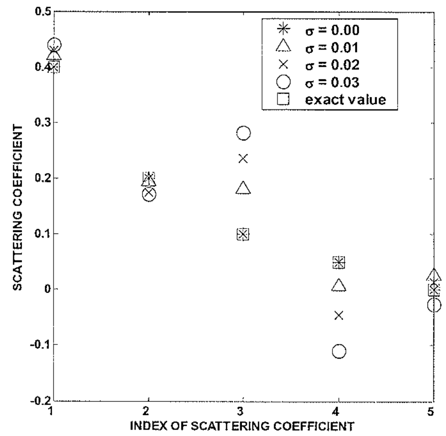
Figure 5. Estimated values for the scattering coefficients. Test Case 3 ($\sigma_{sl} = [0.4,\ 0.2,\ 0.1,\ 0.0,\ 0.0] cm^{-1}$). $L = 0.5\ cm$. $N = 32$.



a) Case 4A ($\sigma_t = 0.5$)



b) Case 4B ($\sigma_t = 0.8$)



c) Case 4C ($\sigma_t = 1.2$)

Figure 6. Estimated values for the scattering coefficients. Test Case 4 ($\sigma_{sl} = [0.4,\ 0.2,\ 0.1,\ 0.05,\ 0.0] cm^{-1}$). $L = 0.5\ cm$. $N = 32$.

Table 3: Estimated values for the scattering (σ_{sl}) and extinction (σ_t) coefficients $\left[cm^{-1}\right]$ using $N = 16$, and $\sigma = 0.0, 0.01, 0.02$ and 0.03. $L = 0.5$ *cm*.

Test case		Estimated values for σ_{sl}								Estimated values for σ_t							
		σ								σ							
		0.00	% error	0.01	% error	0.02	% error	0.03	% error	0.00	% error	0.01	% error	0.02	% error	0.03	% error
1	A	0.4	0.00	0.398	0.52	0.396	0.99	0.394	1.47	0.5	0.00	0.502	0.30	0.504	0.72	0.507	1.35
1	B	0.4	0.00	0.392	1.98	0.385	3.80	0.378	5.49	0.8	0.00	0.799	0.13	0.799	0.14	0.801	0.08
1	C	0.4	0.00	0.386	3.53	0.373	6.82	0.360	9.92	1.2	0.00	1.189	0.90	1.184	1.34	1.183	1.40
2	A	0.4 0.2	0.00 0.00	0.398 0.201	0.51 0.50	0.396 0.203	0.94 1.70	0.394 0.207	1.42 3.36	0.5	0.00	0.502	0.30	0.504	0.70	0.506	1.29
2	B	0.4 0.2	0.00 0.00	0.392 0.200	1.97 0.22	0.385 0.202	3.77 0.81	0.378 0.204	5.42 2.08	0.8	0.00	0.799	0.14	0.799	0.14	0.801	0.06
2	C	0.4 0.2	0.00 0.00	0.387 0.200	3.48 0.22	0.425 0.214	6.72 0.21	0.331 60.19	9.76 0.20	1.2	0.00	1.189	0.90	1.184	1.35	1.183	1.40
3	A	0.4 0.2 0.1	0.00 0.00 0.00	0.398 0.201 0.123	0.51 0.55 23.13	0.396 0.204 0.146	0.95 2.04 45.64	0.394 0.209 0.168	1.59 4.52 67.61	0.5	0.00	0.502	0.30	0.504	0.72	0.517	3.35
3	B	0.4 0.2 0.1	0.00 0.00 0.00	0.392 0.201 0.134	1.98 0.23 0.34	0.385 0.202 0.166	3.80 0.88 66.12	0.378 0.205 0.196	5.47 2.32 96.16	0.8	0.00	0.799	0.12	0.799	0.11	0.800	0.10
3	C	0.4 0.2 0.1	0.00 0.00 0.00	0.386 0.200 0.161	3.55 0.22 61	0.373 0.200 0.213	6.87 0.20 113.3	0.360 0.200 0.259	9.99 0.24 159.0	1.2	0.00	1.190	0.86	1.185	1.28	1.184	1.33
4	A	0.4 0.2 0.1 0.05	0.00 0.00 0.00 0.00	0.398 0.201 0.123 0.072	0.50 0.55 23.13 43.64	0.396 0.204 0.146 0.095	0.94 1.04 45.82 90.41	0.393 0.209 0.167 0.095	1.70 4.27 66.59 148.2	0.5	0.00	0.502	0.31	0.504	0.73	0.515	2.96
4	B	0.4 0.2 0.1 0.05	0.00 0.00 0.00 0.00	0.392 0.201 0.134 0.080	1.97 0.23 34.28 60.32	0.385 0.202 0.166 0.111	3.79 0.88 66.02 121.1	0.378 0.205 0.196 0.142	5.46 2.33 96.14 184	0.8	0.00	0.799	0.12	0.799	0.11	0.801	0.11
4	C	0.4 0.2 0.1 0.05	0.00 0.00 0.00 0.00	0.386 0.200 0.161 0.096	3.55 0.23 60.71 91.62	0.372 0.200 0.213 0.139	6.88 0.20 112.8 178.4	0.36 0.200 0.259 0.181	10.01 0.24 158.5 262.9	1.2	0.00	1.190	0.86	1.185	1.28	1.184	1.32

4. CONCLUSIONS

The matrix based explicit formulation for the inverse radiative transfer problem of scattering and extinction coefficients estimation yielded very encouraging results. It seems to be robust even in the presence of noise in the experimental data.

The discretization of the angular domain seems to provide a regularizing effect. The method deserves further investigation.

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Table 4: Estimated values for the scattering (σ_{sl}) and extinction (σ_t) coefficients $\left[cm^{-1}\right]$ using $N = 14$, and $\sigma = 0.0, 0.01, 0.02$ and 0.03. $L = 0.5$ *cm*.

Test case		Estimated values for σ_{sl}								Estimated values for σ_t							
		σ								σ							
		0.00	% error	0.01	% error	0.02	% error	0.03	% error	0.00	% error	0.01	% error	0.02	% error	0.03	% error
1	A	0.4	0.00	0.398	0.44	0.397	0.74	0.399	0.34	0.5	0.00	0.379	24.27	0.374	25.3	0.369	26.19
1	B	0.41	3.00	0.404	1.04	0.396	0.90	0.389	2.68	0.591	1.63	0.578	26.18	0.561	29.85	0.791	31.35
1	C	0.4	0.00	0.377	4.89	0.400	9.50	0.444	13.89	1.20	0.00	0.889	25.92	0.874	27.15	0.859	28.43
2	A	0.4 0.2	0.00 0.00	0.398 0.194	0.39 2.94	0.397 0.189	0.66 5.62	0.396 0.190	0.92 5.03	0.50	0.00	0.389	24.30	0.373	25.35	0.367	26.53
2	B	0.4 0.2	0.00 0.00	0.392 0.194	2.13 2.95	0.384 0.187	4.11 6.33	0.376 0.181	5.97 9.59	0.80	0.00	0.600	25.01	0.593	25.92	0.583	27.09
2	C	0.4 0.2	0.00 0.00	0.381 0.193	4.83 3.52	0.363 0.185	9.37 7.48	0.345 0.177	13.68 11.54	0.9	25	0..889	25.93	0.874	27.16	0.859	28.44
3	A	0.4 0.2 0.1	0.00 0.00 0.00	0.399 0.194 0.138	0.38 3.18 38.16	0.398 0.188 0.177	0.61 6.05 76.72	0.396 0.185 0.214	1.09 7.62 114.4	0.438	12.5	0.379	24.31	0.373	25.37	0.367	26.57
3	B	0.4 0.2 0.1	0.00 0.00 0.00	0.392 0.194 0.151	2.13 3.04 51.36	0.384 0.187 0.202	4.11 6.60 102.0	0.376 0.180 0.251	5.97 10.02 151.5	0.8	0.00	0.60	25.00	0.593	25.91	0.583	27.09
3	C	0.4 0.2 0.1	0.00 0.00 0.00	0.381 0.193 0.175	4.85 3.57 74.84	0.362 0.185 0.245	9.43 7.63 145.4	0.345 0.177 0.313	13.78 11.81 212.5	1.05	12.5	0.889	25.91	0.874	27.13	0.859	28.42
4	A	0.4 0.2 0.1 0.05	0.00 0.00 0.00 0.00	0.399 0.194 0.138 0.063	0.38 3.17 38.21 25.88	0.398 0.188 0.177 0.073	0.61 6.04 76.87 47.73	0.396 0.185 0.215 0.083	1.10 7.64 114.6 66.01	0.425	15	0.366	26.80	0.361	27.86	0.355	29.07
4	B	0.4 0.2 0.1 0.05	0.00 0.00 0.00 0.00	0.392 0.194 0.151 0.066	2.12 3.03 51.33 31.86	0.384 0.187 0.202 0.079	4.10 6.59 102.0 57.51	0.376 0.18 0.252 0.089	5.95 10.00 151.5 78.17	0.688	14.1	0.588	26.55	0.580	27.46	0.571	28.63
4	C	0.4 0.2 0.1 0.05	0.00 0.00 0.00 0.00	0.380 0.193 0.175 0.071	4.83 3.57 74.74 42.35	0.362 0.185 0.245 0.088	9.40 7.62 145.3 75.02	0.345 0.176 0.313 0.100	13.74 11.81 212.5 99.50	1.037	13.6	0.877	26.94	0.862	28.16	0.847	29.44

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